

An introduction to ludics

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LOCI Symposium:
Rebuilding logic and rethinking language in interaction terms
21 July 2011

Ludics in a few words

Ludics:

- ▶ erases the distinction between syntax and semantics;
- ▶ allows to rebuild logic from the sole notion of interaction.

The basic artifact of ludics is the *design*:

- ▶ designs are abstract representations of linear logic proofs;
- ▶ designs rely on an alternation of polarities in proofs;
- ▶ designs retain only the information relevant for *local* interaction;
- ▶ designs needs not represent correct proofs.

Linear Logic

- ▶ Girard, 80's
- ▶ classical logic: negation is involutive
- ▶ takes cut elimination in sequent calculus seriously
- ▶ drops structural rules

A quick reminder

- ▶ a sequent is a pair of lists: $A_1, \dots, A_n \vdash B_1, \dots, B_p$
- ▶ it “means” $A_1 \wedge \dots \wedge A_n \Rightarrow B_1 \vee \dots \vee B_p$
- ▶ the cut rule is
$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$
- ▶ cut elimination gives proofs without detours, which have good properties
- ▶ up to De Morgan laws, we can restrict to sequents $\vdash B_1, \dots, B_p$ and the cut becomes
$$\frac{\vdash A, B \quad \vdash \neg B, C}{\vdash A, C}$$
- ▶ provable sequents admit cut free proofs

Linear logic: rules

	multiplicative		additive
\wedge :	$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} (\otimes)$		$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B} (\&)$
\vee :	$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} (\wp)$		$\frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2} (\oplus_i)$

Linear logic: rules

$$\frac{}{\vdash X^\perp, X} \text{ (ax)}$$

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\vee :	$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} (\wp)$	$\frac{\vdash \Gamma, A_i}{\vdash \Gamma, A_1 \oplus A_2} (\oplus_i)$

$$\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

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$$\frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \text{ (cut)}$$

$$A, B := X \mid X^\perp \mid A \wp B \mid A \otimes B \mid A \& B \mid A \oplus B$$

$$(A \wp B)^\perp = A^\perp \otimes B^\perp$$

$$(A \& B)^\perp = A^\perp \oplus B^\perp$$

Linear logic: cut elimination

A multiplicative cut (\wp/\otimes):

$$\frac{\frac{\frac{\vdots}{\vdash \Gamma, A} \quad \frac{\vdots}{\vdash \Gamma', B}}{\vdash \Gamma, \Gamma', A \otimes B} (\otimes) \quad \frac{\frac{\vdots}{\vdash \Delta, A^\perp, B^\perp} (\wp)}{\vdash \Delta, A^\perp \wp B^\perp} (\wp)}{\vdash \Gamma, \Gamma', \Delta} (\text{cut})$$

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reduces to

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Linear logic: cut elimination

An additive cut ($\&/\oplus$):

$$\frac{\frac{\frac{\vdots}{\vdash \Gamma, A} \quad \frac{\vdots}{\vdash \Gamma, B}}{\vdash \Gamma, A \& B} (\&) \quad \frac{\frac{\vdots}{\vdash \Delta, A^\perp}}{\vdash \Delta, A^\perp \oplus B^\perp} (\oplus_1)}{\vdash \Gamma, \Delta} (\text{cut})$$

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Linear logic: cut elimination

Identity:

$$\frac{\frac{}{\vdash A, \underline{A^\perp}} \text{(ax)} \quad \vdash \underline{A}, \Gamma \text{ (cut)}}{\vdash A, \Gamma} \text{(cut)}$$

reduces to

$$\vdash \underline{A}, \Gamma$$

Linear logic: cut elimination

Bureaucracy: e.g.,

$$\frac{\frac{\frac{\vdots}{\vdash \Gamma, A, B}}{\vdash \Gamma, A \wp B} (\wp) \quad \frac{\frac{\vdots}{\vdash \Delta, A^\perp \otimes B^\perp, C}}{\vdash \Delta, A^\perp \otimes B^\perp, C \oplus D} (\oplus_1)}{\vdash \Gamma, \Delta, C \oplus D} (\text{cut})$$

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Focusing

Reversibility

The connectives \wp and $\&$ are *reversible*: from the conclusion and active formula, one can recover the premises.

During proof search, one can always perform reversible rules. We thus divide connectors between two classes: \wp and $\&$ are *negative*, and \otimes and \oplus are *positive*.

Positive connectors are *not* reversible but:

Focusing

Every provable sequent admits a focused cut-free proof.

A cut-free proof is *focused* if:

- ▶ each time we decompose a formula using an introduction rule, we focus on its subformulas, as long as they have the same polarity;
- ▶ if a sequent contains a negative formula, we first apply negative rules.

Synthetic connectives: rules

Up to focusing and the distributivity isomorphism $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$, we obtain:

- ▶ one negative (reversible) rule:

$$\frac{\left(\vdash (P_{i,j})_{j \in J_i}, \Gamma \right)_{i \in I}}{\vdash \&_{i \in I} \wp_{j \in J_i} P_{i,j}, \Gamma} (-)$$

- ▶ one positive rule:

$$\frac{\left(\vdash N_{i_0,j}, \Gamma_j \right)_{j \in J_{i_0}}}{\vdash \oplus_{i \in I} \otimes_{j \in J_i} N_{i,j}, \Gamma} (+, i_0)$$

with $\Gamma = \sum_{j \in J_{i_0}} \Gamma_j$.

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Plus axiom and cut.

Synthetic connectives: cut elimination

$$\frac{\frac{\left(\begin{array}{c} \pi_j \\ \vdots \\ \vdots \\ \vdots \end{array} \right)_{j \in J_{i_0}}}{\vdash \bigoplus_{i \in I} \bigotimes_{j \in J_i} P_{i,j}^\perp, \Gamma} (+, i_0)}{\vdash \bigoplus_{i \in I} \bigotimes_{j \in J_i} P_{i,j}^\perp, \Gamma} \quad \frac{\frac{\left(\begin{array}{c} \rho_{i,j} \\ \vdots \\ \vdots \\ \vdots \end{array} \right)_{i \in I}}{\vdash \bigwedge_{i \in I} \bigvee_{j \in J_i} P_{i,j}, \Delta} (-)}{\vdash \bigoplus_{i \in I} \bigotimes_{j \in J_i} P_{i,j}^\perp, \Gamma} (\text{cut})}{\vdash \Gamma, \Delta}$$

Synthetic connectives: cut elimination

$$\frac{
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 }{
 \vdash \bigoplus_{i \in I} \bigotimes_{j \in J_i} P_{i, j}^\perp, \Gamma
 }
 \quad
 \frac{
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 }{
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reduces to

$$\frac{\left(\begin{array}{c} \pi_j \\ \vdots \\ \vdots \end{array} \right)_{j \in J_{i_0}} \quad \rho_{i_0,j}}{\vdash \bigwedge_{j \in J_{i_0}} P_{i_0,j}, \Delta} \quad \vdash \bigvee_{j \in J_{i_0}} P_{i_0,j}, \Delta}{\vdash (\Gamma_j)_{j \in J_{i_0}}, \Delta} (\text{cut}) \times \#J_{i_0}$$

Loci

Ludics founds logic on the interaction between proofs:
cut-elimination between A and A^\perp .

To enable this dialogue without preconception:

- ▶ Ludics forgets about the meaning of formulas.
Sequents only retain information on the location of subformulas: the *locus*.
- ▶ It introduces a generic “dummy” proof: the *daimon*.
The essential point of interaction is that both parties should reach an agreement: one must give up, using the daimon.

Definition

An address (or locus) is a finite list of natural numbers.

A sequent is a pair $\Lambda \vdash \Delta$ where Λ holds at most one formula.

If $\Lambda = \emptyset$ the sequent is positive, otherwise it is negative.

Designs

... as abstract proof trees (dessins)

daimon

$$\frac{}{\vdash \Delta} (\boxtimes)$$

negative rule

$$\frac{(\vdash (\xi i)_{i \in I}, \Delta_i)_{i \in \mathcal{N}}}{\xi \vdash \Delta} (-, \xi, \mathcal{N})$$

where $\mathcal{N} \subseteq \mathfrak{P}_f(\mathbf{N})$ and each $\Delta_i \subseteq \Delta$.

positive rule

$$\frac{(\xi i \vdash \Delta_i)_{i \in I}}{\vdash \xi, \Delta} (+, \xi, I)$$

where I is finite, $\bigcup \Delta_i \subseteq \Delta$, and $\Delta_i \cap \Delta_j = \emptyset$ for all $i \neq j$.

Proofs as designs

$$\frac{\frac{\frac{\vdots}{\vdash P, Q, S} \quad \frac{\vdots}{\vdash R, S}}{\vdash (P \wp Q) \& R, S} (-)}{\vdash T \oplus ((P \wp Q) \& R) \oplus U, S} (+, \{2\})$$

becomes

$$\frac{\frac{\frac{\vdots}{\vdash \xi_{21}, \xi_{22}, \sigma} \quad \frac{\vdots}{\vdash \xi_{23}, \sigma}}{\vdash \xi, \sigma} (-, \xi_2, \{\{1, 2\}, \{3\}\})}{\vdash \xi, \sigma} (+, \xi, \{2\})$$

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- ▶ There is not even an axiom rule: see later.
- ▶ Designs as dessins (trees) actually retain irrelevant information about the context of rules: compare

$$\frac{\frac{\frac{}{\vdash \xi_{12}} (\text{✠})}{\xi_1 \vdash \sigma} (-, \xi_1, \{\{2\}\})}{\vdash \xi, \sigma} (+, \xi, \{1\}) \quad \text{with} \quad \frac{\frac{\frac{}{\vdash \xi_{12}} (\text{✠})}{\xi_1 \vdash \sigma} (-, \xi_1, \{\{2\}\})}{\vdash \xi, \sigma} (+, \xi, \{1\})$$

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One can introduce a further level of abstraction to fix this: designs as strategies (desseins).

Intuitively: desseins = sets of branches in a dessin.

Interaction: cut nets

Definition

A cut net is a non empty set of designs s.t.:

- ▶ addresses in conclusions are either disjoint or identical;
- ▶ each address appears in at most two conclusions, and then with opposite polarities: this is a *cut*;
- ▶ the graph with conclusions as vertices and cuts as arrows is connected and acyclic.

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In particular there is exactly one design without a cut on the left: its conclusion is the main sequent and its last rule the main rule.

Interaction: cut elimination as normalization

The case of closed nets: all addresses are cuts

The main design D is then necessarily positive.

- ▶ The main rule is (\boxplus) : normalization immediately ends and results in \boxplus .
- ▶ The main rule is $(+, \xi, I)$: then ξ is a cut, with the negative address of another design E , whose last rule is $(-, \xi, \mathcal{N})$.
 - ▶ if $I \notin \mathcal{N}$, normalization fails;
 - ▶ otherwise, for all $i \in I$, we consider the subdesign D_i of D with conclusion $(\xi i \vdash \dots)$, and the subdesign E' of E with conclusion $(\vdash \xi i, \dots)$: we replace D with the D_i 's and E with E' . We normalize the net obtained as the component of E' .

The general case

When none of the above cases applies, we normalize above the main rule (*cf.* commutative cuts in sequent calculus).

Example

Start from a net made of two designs:

$$\frac{\frac{\frac{\vdots}{\xi 1 \vdash} \quad \frac{\vdots}{\xi 2 \vdash \sigma 3 1}}{\vdash \xi, \sigma 3 1} (+, \xi, \{1, 2\})}{\frac{\vdots}{\sigma 3 \vdash \xi} (-, \sigma 3, \{\{1\}\})}{\vdash \xi, \sigma} \quad \frac{\vdots}{\sigma 7 \vdash} (+, \sigma, \{3, 7\})$$

$$\frac{\frac{\vdots}{\vdash \xi 0, \tau} \quad \frac{\vdots}{\vdash \xi 1, \xi 2, \tau} \quad \frac{\vdots}{\vdash \xi 3, \tau}}{\xi \vdash \tau} (-, \xi, \{\{0\}, \{1, 2\}, \{3\}\})$$

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We reached a genuine cut.

Example

It remains to normalize a cut net made of three designs:

$$\frac{\frac{\begin{array}{c} \vdots \\ \xi 1 \vdash \end{array} \quad \frac{\begin{array}{c} \vdots \\ \xi 2 \vdash \sigma 3 1 \end{array}}{\sigma 3 \vdash} \quad (-, \sigma 3, \{\{1\}\})}{\vdash \sigma} \quad \frac{\begin{array}{c} \vdots \\ \sigma 7 \vdash \end{array} \quad (+, \sigma, \{3, 7\})}{\vdash \sigma}$$

$$\frac{\vdots}{\vdash \xi 1, \xi 2, \tau}$$

Fax

There are no axioms, because there are no formulas.

Instead there is a generic η -expansion, given by the fax design $\mathfrak{F}_{\xi, \xi'}$:

$$\frac{\begin{array}{c} \mathfrak{F}_{\xi' i, \xi i} \\ \vdots \\ \dots \quad \frac{\xi' i \vdash \xi i}{\vdash \xi', (\xi i)_{i \in I}} \quad \dots \quad (+, \xi', I) \\ \dots \end{array}}{\xi \vdash \xi'} \quad (-, \xi, \mathfrak{P}_f(\mathbf{N}))$$

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$$\frac{\begin{array}{c} \mathfrak{F}_{\xi'1, \xi1} \\ \vdots \\ \dots \quad \frac{\xi'1 \vdash \xi1}{\vdash \xi', (\xi i)_{i \in I}} \quad \dots \end{array}}{\xi \vdash \xi'} \quad (+, \xi', I) \quad \dots \quad (-, \xi, \mathfrak{P}_f(\mathbf{N}))$$

The axiom $P \oplus Q \vdash P \oplus Q$ becomes:

$$\frac{\begin{array}{c} \mathfrak{F}_{\xi'1, \xi1} \\ \vdots \\ \frac{\xi'1 \vdash \xi1}{\vdash \xi1, \xi'} \quad (+, \xi', \{1\}) \end{array} \quad \begin{array}{c} \mathfrak{F}_{\xi'1, \xi1} \\ \vdots \\ \frac{\xi'2 \vdash \xi2}{\vdash \xi2, \xi'} \quad (+, \xi', \{2\}) \end{array}}{\xi \vdash \xi'} \quad (-, \xi, \{\{1\}, \{2\}\})$$

Fax

There are no axioms, because there are no formulas.

Instead there is a generic η -expansion, given by the fax design $\mathfrak{F}_{\xi, \xi'}$:

$$\frac{\begin{array}{c} \mathfrak{F}_{\xi' i, \xi i} \\ \vdots \\ \dots \quad \frac{\xi' i \vdash \xi i}{\vdash \xi', (\xi i)_{i \in I}} \quad \dots \quad (+, \xi', I) \\ \dots \end{array}}{\xi \vdash \xi'} \quad (-, \xi, \mathfrak{P}_f(\mathbf{N}))$$

Normalizing a design D of conclusion $\xi' \vdash \Gamma$ with $\mathfrak{F}_{\xi, \xi'}$ results in a relocalized design D' , with conclusion $\xi \vdash \Gamma$.

Rebuilding logic: orthogonality

Definition

Let D be a design with conclusion $\Lambda \vdash \Gamma$ and for all $\xi \in \Lambda \cup \Gamma$, let E_ξ be a designs of conclusion $\vdash \xi$ or $\xi \vdash$ so that $N = \{D\} \cup \{E_\xi \mid \xi \in \Lambda \cup \Gamma\}$ is a closed cut net. We say D is orthogonal to (E_ξ) if N normalizes to the daimon.

Rebuilding logic: behaviours

Definition

Let \mathbf{D} be a set of designs with the same conclusion: we write $\mathbf{D}^{\perp\perp}$ for its bidual.

We say \mathbf{D} is a behaviour if $\mathbf{D} = \mathbf{D}^{\perp\perp}$.

Rebuilding logic: behaviours

Definition

Let \mathbf{D} be a set of designs with the same conclusion: we write $\mathbf{D}^{\perp\perp}$ for its bidual.

We say \mathbf{D} is a behaviour if $\mathbf{D} = \mathbf{D}^{\perp\perp}$.

Behaviours are the ludics counterpart of formulas.

Rebuilding logic: additives

- ▶ Any intersection of behaviours is a behaviour.
- ▶ It does not necessarily hold for union: write $\sqcup \mathbf{D}_i = (\cup \mathbf{D}_i)^{\perp\perp}$.
- ▶ If $\mathbf{D}_1 \cap \mathbf{D}_2 = \emptyset$, $\mathbf{D}_1 \sqcup \mathbf{D}_2 = \mathbf{D}_1 \cup \mathbf{D}_2$.

Fact

\cap and \sqcup provide *locative* interpretations of $\&$ and \oplus .

To recover the usual connectives, we should introduce some more structure.

Rebuilding logic: multiplicatives

The basic idea is to introduce a binary operation on positive designs:

if the first (positive) actions of D and D' are I and J , we form a new design $D \odot D'$ with first action $I \cup J$, and branches selected among those of D and D' .

Fact

Several choices for \odot are possible, with interesting properties.

Setting $\mathbf{D} \otimes \mathbf{D}' = \{D \odot D' \mid D \in \mathbf{D}, D' \in \mathbf{D}'\}^{\perp\perp}$ provides a locative interpretation of tensor.

We recover \wp by duality.

What is missing from this talk?

Almost everything :-)

- ▶ the good notion of designs (desseins);
- ▶ beautiful theorems (associativity, separation, stability, . . .);
- ▶ the notion of truth;
- ▶ completeness theorems;
- ▶ *etc.*

(...)